SEM – II Paper: CC – III

Study material on Magnetostatics Part 1

Dr. Dipika Saha

The Magnetic Field

- We know that a stationery charges gets up a electric field E in the space surrounding it and this electric field exerts a force $\mathbf{F}=q_0\mathbf{E}$ on the test charge q_0 placed in magnetic field.
- Similarly, we can describe the interaction of moving charges that, a moving charge exert a magnetic field in the space surrounding it and this magnetic field exert a force on the moving charge.
- Like electric field, magnetic field is also a vector quantity and is represented by symbol **B**
- Like electric field force which depend on the magnitude of charge and electric field, magnetic force is proportional to the magnitude of charge and the strength of magnetic field.
- Apart from its dependence on magnitude of charge and magnetic field strength, magnetic force also depends on velocity of the particle.
- The magnitude of magnetic force increases with increase in speed of charged particle.
- Direction of magnetic force depends on direction of magnetic field B and velocity **v** of the charged particle.
- The direction of magnetic force is not along the direction of magnetic field but direction of force is always perpendicular to direction of both magnetic field **B** and velocity **v.**
- Test charge of magnitude q_0 is moving with velocity **v** through a point P in magnetic field **B** experience a deflecting force **F** defined by
  $\mathbf{F}=q\mathbf{v} \times \mathbf{B}$ **……(1)**
- This force on charged particle is perpendicular to the plane formed by **v** and **B** and its direction is determined right hand thumb rule.
- When moving charge is positive the direction of force **F** is the direction of advancement of right hand screw whose axis is perpendicular to the plane formed by **v** and **B**.
- Direction of force would be opposite to the direction of advancement of right hand screw for negative charge moving in same direction.
- Magnitude of force on charged particle is
  $F=q_0vB\sin\theta$ ……(2) where θ is the angle between v and B.
- If **v** and **B** are at right angle to each other i.e. θ=90 then force acting on the particle would be maximum and is given by
  $F_{max} = q_0vB$ ……(3)
- When $\theta=180^0$ or θ=0 i.e. v is parallel or anti parallel to B then force acting on the particle would be zero.
- Again from equation 2 if the velocity of the particle in the magnetic field is zero i.e., particle is stationery in magnetic field then it does not experience any force.
- SI unit of strength of magnetic field is Tesla (T). It can be defined as follows
  $B=F/qv\sin\theta$
  for F=1N, q=1C and v=1m/s and $\theta=90^0$
  $1T=1NA^{-1}m^{-1}$
  Thus if a charge of 1C when moving with velocity of 1 m/s along the direction

perpendicular to the magnetic field experiences a force of 1N then magnitude of field at that point is equal to 1 Tesla (1T).

- Another SI unit of magnetic field is Weber/m^2. Thus
  1 Wb-m^{-2}=1T=1N$A^{-1}m^{-1}$
  In CGS system, the magnetic field is expressed in 'gauss'. And 1T= 10^4 gauss.
  Dimension formula of magnetic field (B) is [$MT^{-2}A^{-1}$]

Lorentz Force

- We know that force acting on any charge of magnitude q moving with velocity **v** inside the magnetic field B is given by
  F = q(**v** X **B**)
  and this is the magnetic force on charge q due to its motion inside magnetic field.
- If both electric field E and magnetic field B are present i.e., when a charged particle moves through a region of space where both electric field and magnetic field are present both field exert a force on the particle and the total force on the particle is equal to the vector sum of the electric field and magnetic field force.
  F = q**E** + q(**v** X **B**) (4)
- This force in equation (4) is known as Lorentz Force.
- An important point to note that magnetic field is not doing any work on the charged particle as it always act in perpendicular direction to the motion of the charge.

Biot-Savart Law

We know that electric current or moving charges are source of magnetic field. A Small current carrying conductor of length **dl** carrying current I is a elementary source of magnetic field. The force on another similar conductor can be expressed conveniently in terms of magnetic field **dB** due to the first.
The dependence of magnetic field **dB** on current I, on size and orientation of the length element **dl** and on distance **r** was first guessed by Biot and Savart.

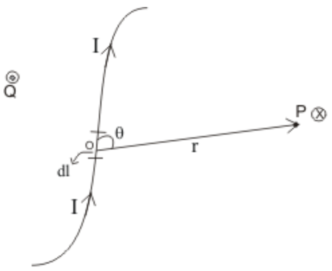


Figure 1. Field at point P is perpendicular to the plane of paper pointing into it

The magnitude of the magnetic field **dB** at a distance **r** from a current element **dl** carrying current I is found to be proportional to I, to the length dl, the sine of angle θ between the line element dl and radius vector **r** and inversely proportional to the square of the distance |**r**|. The direction of the magnetic Field is perpendicular to the line element **dl** as well as radius **r.**
The magnitude of magnetic field is

$$dB = \left(\frac{\mu_0}{4\pi}\right)\frac{I\,|\,\mathbf{dl}\,|\sin\theta}{r^2}$$

Mathematically, Field **dB** is written as

$$d\boldsymbol{B} = \left(\frac{\mu_0}{4\pi}\right) I \frac{d\boldsymbol{l} \times \hat{\boldsymbol{r}}}{r^2}$$

or,

$$d\boldsymbol{B} = \left(\frac{\mu_0}{4\pi}\right) I \frac{d\boldsymbol{l} \times \boldsymbol{r}^3}{r^2}$$

In the figure, considering that the line element **dl** and radius vector **r** connecting mid point of line element to the field point P at which field is to be found are in the plane of the paper. From the above equation, we expect magnetic field to be perpendicular to both **dl** and **r**. Thus direction of **dB** is the direction of advance of right hand screw whose axis is perpendicular to the plane formed by **dl** and **r** and which is rotated from **dl** to **r** (right hand screw rule of vector product). Thus in figure, **dB** at point P is perpendicular directed in downwards represented by the symbol (x) and point Q field is directed in upward direction represented by the symbol (•). The resultant field at point P due to whole conductor can be found by integrating dB over the length of the conductor i.e. **B** = ∫d**B**

Comparison between Coulomb's law and Biot-Savart law

- Both the electric and magnetic field depends inversely on square of distance between the source and field point. Both of them are long range forces.
- Charge element dq producing electric field is a scalar whereas the current element I**dl** is a vector quantity having direction same as that of flow of current.
- According to coulomb's law, the magnitude of electric field at any point P depends only on the distance of the charge element from any point P. According to Biot-savart law, the direction of magnetic field is perpendicular to the current element as well as to the line joining the current element to the point P.
- Both electric field and magnetic field are proportional to the source strength namely charge and current element respectively. This linearity makes it simple to find the field due to more complicated distribution of charge and current by superposing those due to elementary changes and current elements

Applications of Biot-Savart law

(i) Magnetic Field due to steady current in an infinitely long straight wire:

Consider a straight infinitely long wire carrying a steady current I. We want to calculate magnetic field at a point P at a distance R from the wire as shown below in figure

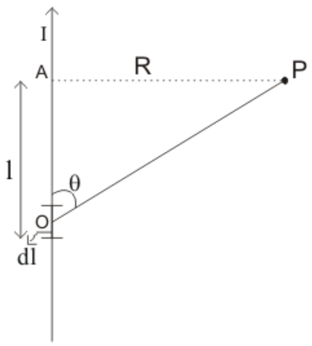


Fig. 1 Infinitely long wire carrying current I

From Biot-Savart law, magnetic field d**B** due to small current element of the wire at point O at a distance |**r**|=r from point P is

$$d\boldsymbol{B} = \left(\frac{\mu_0}{4\pi}\right) \mathrm{I} \frac{\mathbf{d}\boldsymbol{l} \times \boldsymbol{r}^3}{\mathbf{r}^2}$$

(1)

since current element Id**l** and vector **r** makes an angle θ with each other, the magnitude of the product d**l** x **r** is dlrsinθ and is directed perpendicular to both dl and r vector

If we take the current element along the y-axis and AP along the x-axis then magnetic field B will be along z-axis therefore we can write

$$d\mathbf{B} = \left(\frac{\mu_0}{4\pi}\right) \frac{I\,|\,\mathbf{dl}\,|\sin\theta}{r^2}\mathbf{k}$$

(2)

where **k** is the unit vector along z-axis. we will now express sinθ and r in terms of R which is fixed distance for any point in space and *l* which describes the position of current element on the infinitely long wire. From figure 1 we have

$$\sin\theta = \frac{R}{(R^2 + l^2)^{1/2}}$$

and r = $(R^2 + l^2)^{1/2}$

Putting these values in the equation (2) we find

$$d\mathbf{B} = \left(\frac{\mu_0}{4\pi}\right) \frac{IdlR}{(R^2 + l^2)^{3/2}}\mathbf{k}$$

(3)

To find the field due to entire straight wire carrying wire ,we would have to integrate equation (6) **B** = ∫d**B or**

$$\mathbf{B} = \frac{\mu_0 IR\mathbf{k}}{4\pi} \int_{-\infty}^{\infty} \frac{dl}{(R^2 + l^2)^{3/2}}$$

(4)

To evaluate the integral on the RHS substitute *l* = RtanΦ and d*l* = Rsec^{2}Φ dΦ

Therefore

$$\mathbf{B} = \frac{\mu_0 I}{4\pi R}\mathbf{k} \int_{-\pi/2}^{\pi/2} \cos\phi\, d\phi$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi R}\mathbf{k}$$

From the above equation, we noticed that

- Magnetic field is proportional to the current I
- It is inversely proportional to the distance R and
- Magnetic field is in the direction perpendicular to the straight wire and vector **AP** = **R**

- The magnetic line of force near a linear current carrying wire are concentric circles around the conductor in a plane perpendicular to the wire
- Hence the direction of field B at point P at a distance R from wire, will be along the tangent drawn on a circle of radius R around the conductor as shown below in figure 2

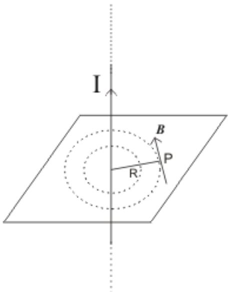
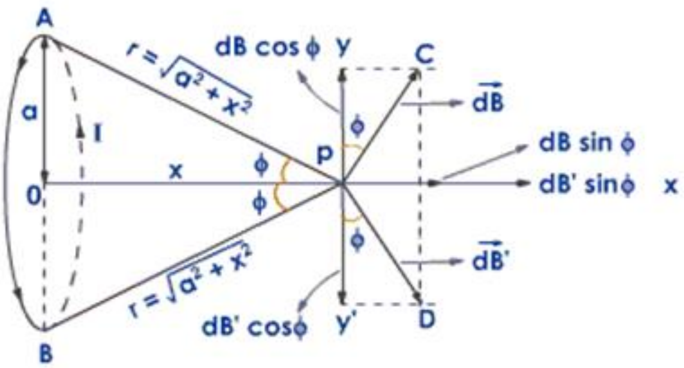


Fig.2: Direction of magnetic field at point P

- Direction of B can be found by right hand thumb rule i.e. grasp the wire with right hand, the thumb pointing in direction of current, the finger will curl around the wire in the direction of **B**
- The magnetic field lines are circular closed curve around the wire.

Magnetic Field along axis of a circular current carrying coil:



Consider a circular coil having radius a and centre O from which current I flows in anticlockwise direction. The coil is placed at YZ plane so that the centre of the coil coincide along X-axis. P be the any point at a distance x from the centre of the coil where we have to calculate the magnetic field. Let dl be the small current carrying element at any point A at a distance r from the point P where r = $\sqrt{a^2 + x^2}$ and the angle between r and dl is 90°. Then from biot-savart law, the magnetic field due to current carrying element dl is

$$dB = \frac{\mu_\circ}{4\pi}\frac{Idlsin\theta}{r^2} = \frac{\mu_\circ}{4\pi}\frac{Idlsin90}{r^2} = \frac{\mu_\circ}{4\pi}\frac{Idl}{r^2}$$

And the direction of magnetic field is perpendicular to the plane containing dl and r. So, the magnetic field dB has two components dBcosθ along the Y-axis and dB sinθ along the Y-axis. Similarly, consider another current carrying element dl′ which is diametrically opposite to the point A. The magnetic field due to this current carrying element dB′ also has two components dB′cosθ along the Y-axis and dB′sinθ along the X-axis.

Here both dBcosθ and dB′cosθ are equal in magnitude and opposite in direction. So, they cancel each other. Similarly, the components dBsinθ and dB′sinθ are equal in magnitude and in same direction so they add up.

So, total magnetic field due to the circular current carrying coil at the axis is

$$B = \int_0^{2\pi a} dB sin\theta = \int_0^{2\pi a} \frac{\mu_\circ}{4\pi}\frac{Idl}{r^2}\frac{a}{r}$$

$$since\ sin\theta = \frac{a}{r} B = \int_0^{2\pi a} \frac{\mu_\circ}{4\pi}\frac{Idl}{(x^2+a^2)}\frac{a}{(x^2+a^2)^{\frac{1}{2}}} = \frac{\mu_\circ}{4\pi}\frac{Ia}{(x^2+a^2)^{\frac{3}{2}}}\int_0^{2\pi a} dl$$

$$B = \frac{\mu_\circ}{4\pi}\frac{Ia}{(x^2+a^2)^{\frac{3}{2}}}2\pi a$$

$$B = \frac{\mu_\circ}{2}\frac{Ia^2}{(x^2+a^2)^{\frac{3}{2}}}\ Tesla$$

This is the expression for magnetic field due to circular current carrying coil along its axis. If the coil having N number of turns then magnetic field along its axis is

$$B = \frac{\mu_\circ}{2}\frac{INa^2}{(x^2+a^2)^{\frac{3}{2}}}\ Tesla$$

Direction of B: Direction of magnetic field at a point on the axis of circular coil is along the axis and its orientation can be obtained by using right hand thumb rule .If the fingers are curled along the current, the stretched thumb will point towards the magnetic field and Magnetic field will be out of the page for anti-clockwise current and into the page for clockwise direction.
(i) At the centre of the coil x = 0 so,

$$B_{centre} = \frac{\mu_0}{2}\frac{Ia^2}{a^3} = \frac{\mu_0}{2}\frac{I}{a}$$

(ii) At the point far away from the centre x>>>a
In this case a in the denominator can be neglected hence

$$B = \frac{\mu_0}{2}\frac{Ia^2}{x^3}$$

For coil having N number of turns

$$B = \frac{\mu_0}{2}\frac{NIa^2}{x^3}$$

If the area A of the coil is πR^2 then

$$B = \frac{\mu_0}{2}\frac{NIA}{\pi x^3}$$

m=NIA represents the magnetic moment of the current coil then we have

$$B = \frac{\mu_0}{2}\frac{m}{\pi x^3}$$

Current loop as a magnetic dipole

Ampere found that the distribution of magnetic lines of force around a finite current carrying solenoid is similar to that produced by a bar magnet. This is evident from the fact that a compass needle when moved around these two bodies show similar deflections. After noting the close resemblance between these two, Ampere demonstrated that a simple current loop behaves like a bar magnet and put forward that all the magnetic phenomena is due to circulating electric current. This is Ampere’s hypothesis.

The magnetic induction at a point along the axis of a circular coil carrying current is

$$B = \frac{\mu_\circ}{2} \frac{INa^2}{(x^2 + a^2)^{\frac{3}{2}}} \ Tesla$$

The direction of this magnetic field is along the axis and is given by right hand rule. For points which are far away from the centre of the coil, $x >> a$, a^2 is small and it is neglected. Hence for such points,

$$B = \frac{\mu_0}{2} \frac{NIa^2}{x^3}$$

If we consider a circular loop, n = 1, its area A = πa^2

$$B = \frac{\mu_0}{2} \frac{IA}{\pi x^3} \qquad \text{.....(1)}$$

The magnetic induction at a point along the axial line of a short bar magnet is

$$B = \frac{\mu_0}{4\pi} \frac{2M}{x^3}$$

$$B = \frac{\mu_0}{2\pi} \frac{M}{x^3} \qquad \text{......(2)}$$

Comparing equations (1) and (2), we find that

M = IA ....(3)

Hence a current loop is equivalent to a **magnetic dipole of moment M = IA**

The magnetic moment of a current loop is defined as the product of the current and the loop area. Its direction is perpendicular to the plane of the loop.

Ampere's circuital law:

Ampere's circuital law in magnetism is analogous to gauss's law in electrostatics. This law is also used to calculate the magnetic field due to any given current distribution. This law states that, "the line integral of resultant magnetic field along a closed plane curve is equal to μ_0 time the total current crossing the area bounded by the closed curve provided the magnetic field inside the loop remains constant".

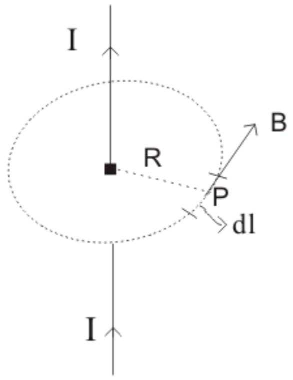
Thus

$$\oint \mathbf{B.dl} = \mu_0 I_{enc}$$

where μ_0 is the permeability of free space and I_{enc} is the net current enclosed by the loop.

Proof of Ampere's Law

Consider a long straight conductor carrying current I perpendicular to the page in upward direction as shown below in the figure



From Biot Savart law, the magnetic field at any point P which is at a distance R from the conductor is given by

$$B = \frac{\mu_0 I}{2\pi R}$$

Direction of magnetic Field at point P is along the tangent to the circle of radius R. For every point on the circle magnetic field has same magnitude as given by

$$B = \frac{\mu_0 I}{2\pi R}$$

And field is tangent to the circle at each point. The line integral of B around the circle is

$$\oint \mathbf{B.dl} = \oint \frac{\mu_0 I}{2\pi R} dl \quad = \frac{\mu_0 I}{2\pi R} \oint dl$$

since ∫dl=2πR i.e., circumference of the circle so,

$$\oint \mathbf{B.dl} = \mu_0 I$$

This is the same result as stated by Ampere law. This ampere's law is true for any assembly of currents and for any closed curve though we have proved the result using a circular Amperian loop. If the wire lies outside the Amperian loop, the line integral of the field of that wire will be zero.

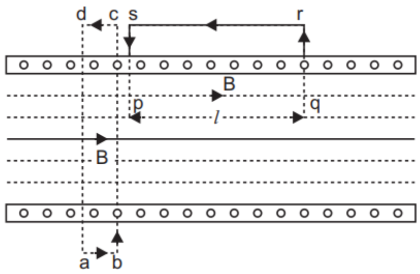
$$\oint \mathbf{B.dl} = 0$$

This does not necessarily mean that **B**=0 everywhere along the path, but only that no current is linked by the path. While choosing the path for integration, we must keep in mind that point at which field is to be determined must lie on the path and the path must have enough symmetry so that the integral can be evaluated.

Applications of Ampere's Circuital law

(i) Magnetic Field Due to a Current Carrying Long Solenoid:

A solenoid is a long wire wound in the form of a close-packed helix, carrying current. To construct a solenoid a large number of closely packed turns of insulated copper wire are wound on a cylindrical tube of card-board or china clay. When an electric current is passed through the solenoid, a magnetic field is produced within the solenoid. If the solenoid is long and the successive insulated copper turns have no gaps, then the magnetic field within the solenoid is uniform; with practically no magnetic field outside it. The reason is that the solenoid may be supposed to be formed of a large number of circular current elements. The magnetic field due to a circular loop is along its axis and the current in upper and lower straight parts of solenoid is equal and opposite. Due to this the magnetic field in a direction perpendicular to the axis of solenoid is zero and so the resultant magnetic field is along the axis of the solenoid. If there are 'n' number of turns per metre length of solenoid and I amperes is the current flowing, then magnetic field at axis of long solenoid is $B = \mu_0 nI$



If there are N turns in length *l* of wire, then $n = N/l$ *or* $B = \mu_0 NI/l$
Derivation: Consider a symmetrical long solenoid having number of turns per unit length equal to n. Let I be the current flowing in the solenoid, then by right hand rule, the magnetic field is parallel to the axis of the solenoid.
Field outside the solenoid: Consider a closed path abcd. Applying Ampere's law to this path

$$\oint \vec{B} \bullet \vec{dl} = \mu \times 0$$

(since net current enclosed by path is zero)
As $dl \neq 0 \therefore B = 0$
This means that the magnetic field outside the solenoid is zero.
Field Inside the solenoid: Consider a closed path pqrs. The line integral of magnetic field vector B along path pqrs is

$$\oint_{pqrs} \vec{B} \bullet \vec{dl} = \int_{pq} \vec{B} \bullet \vec{dl} + \int_{qr} \vec{B} \bullet \vec{dl} + \int_{rs} \vec{B} \bullet \vec{dl} + \int_{sp} \vec{B} \bullet \vec{dl} \quad ...(i)$$

For path *pq*, $\vec{B}$ and $\vec{dl}$ are along the same direction,

$$\therefore \qquad \int_{pq} \vec{B} \bullet \vec{dl} = \int B\, dl = Bl \qquad (pq = l \text{ say})$$

For paths *qr* and *sp*, $\vec{B}$ and $d\vec{l}$ are mutually perpendicular.

$$\therefore \qquad \int_{qr} \vec{B} \bullet \vec{dl} = \int_{sp} \vec{B} \bullet d\vec{l} = \int B\, dl \cos 90° = 0$$

For path *rs*, $B = 0$ (since field is zero outside a solenoid)

$$\therefore \qquad \int_{rs} \vec{B} \bullet \vec{dl} = 0$$

In view of these, equation (*i*) gives

$$\oint_{pqrs} \vec{B} \bullet \vec{dl} = \int_{pq} \vec{B} \bullet \vec{dl} = Bl \qquad ...(ii)$$

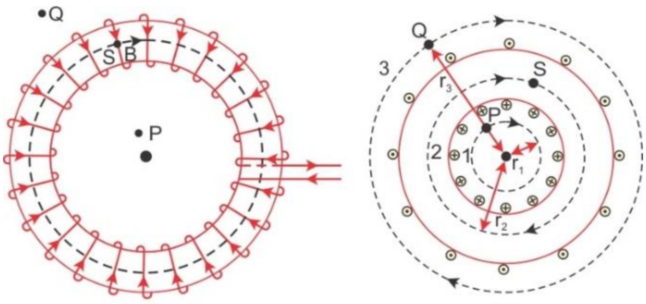
By Ampere's law $\oint \vec{B} \bullet \vec{dl} = \mu_0 \times$ net current enclosed by path

$$\therefore \qquad Bl = \mu_0 (nl\, I) \qquad \therefore \quad B = \mu_0\, nI$$

(ii) Magnetic field along the axis of toroid:

Let n be the number of turns per unit length of toroid and I be the current flowing through it. A magnetic field of constant magnitude is set up inside the turns of toroid in the form of concentric circular magnetic field lines. The direction of the magnetic field at a point is given by the tangent to the magnetic field line at that point.

We draw three circular amperian loops, 1, 2 and 3 of radii r_1, r_2 and r_3 to be traversed in clockwise direction as shown by dashed circles in Fig (b), so that the points P, S and Q may lie on them. The circular area bounded by loops 2 and 3, both cut the toroid. Each turn of current carrying wire is cut once by the loop 2 and twice by the loop 3. Let B_1 be the magnitude of magnetic field along loop 1. Line integral of magnetic field B_1 along the loop 1 is given by



(a) (b)

$$\oint_{loop1} \overline{B_1}.\overline{d\ell} = \oint_{loop1} B_1 d\ell \cos 0^\circ = B_1\ 2\pi r_1$$

Loop 1 encloses no current.

According to Ampere's circuital law

$$\oint_{loop1} \overline{B_1}.\overline{d\ell} = \mu_0 \times \text{current enclosed by loop1} = \mu_0 \times 0 = 0$$

or

$B_1\ 2\pi r_1 = 0$ or $B_1 = 0$

Let B_3 be the magnitude of magnetic field along the loop3.

The line integral of magnetic field B_3 and the loop3 is

$$\oint_{loop3} \overline{B_3}.\overline{d\ell} = \oint_{loop3} B_3 d\ell \cos 0^\circ = B_3\ 2\pi r_3$$

From the sectional cut as shown in Fig.(b),we note that the current coming out of the plane paper is cancelled exactly by the current going into it.

Therefore,the total current enclosed by loop3 is zero.

According to Ampere's circuital law

$$\oint_{loop3} \overline{B_3}.\overline{dl} = \mu_0 \times \text{total current through loop3}$$

or

$B_3 2 \pi r_3 = \mu_0 \times 0 = 0$

or $B_3 = 0$

Let B the magnitude of magnetic field along the loop2.

Line integral of magnetic field along the loop is

$$\oint_{loop2} \vec{B}.\overline{d\ell} = B \times 2 \pi r_2$$

Current enclosed by the loop2 = number of turns× current in each turn $= 2 \pi r_2 n \times I$

According to Ampere's circuital law

$$\oint_{loop2} \vec{B}.\overline{d\ell} = \mu_0 \times \text{total current}$$

or

$B\ 2 \pi r_2 = \mu_0 \times 2 \pi r_2 n I$

or

$B = \mu_0 n I$